

Knowledge structure, knowledge granulation and knowledge distance in a knowledge base

Yuhua Qian^{a,b,c}, Jiye Liang^{a,b,*}, Chuangyin Dang^c

^a*Key Laboratory of Computational Intelligence and Chinese Information Processing of Ministry of Education, Taiyuan, 030006, China*

^b*School of Computer and Information Technology, Shanxi University, Taiyuan, 030006, China*

^c*Department of Manufacturing Engineering and Engineering Management, City University of Hong Kong, Hong Kong*

Abstract

One of the strengths of rough set theory is the fact that an unknown target concept can be approximately characterized by existing knowledge structures in a knowledge base. Knowledge structures in knowledge bases have two categories: complete and incomplete. In this paper, through uniformly expressing these two kinds of knowledge structures, we first address four operators on a knowledge base, which are adequate for generating new knowledge structures through using known knowledge structures. Then, an axiom definition of knowledge granulation in knowledge bases is presented, under which some existing knowledge granulations become its special forms. Finally, we introduce the concept of a knowledge distance for calculating the difference between two knowledge structures in the same knowledge base. Noting that the knowledge distance satisfies the three properties of a distance space on all knowledge structures induced by a given universe. These results will be very helpful for knowledge discovery from knowledge bases and significant for establishing a framework of granular computing in knowledge bases.

Key words: Rough set theory; Granular computing; Knowledge bases; Knowledge granulation; Knowledge distance

* Corresponding author.

Email addresses: jinchengqyh@126.com (Yuhua Qian), ljiy@sxu.edu.cn (Jiye Liang), mec-dang@cityu.edu.hk (Chuangyin Dang).

1. Introduction

In 1982, Pawlak proposed a new set theory, so-called rough set theory [27, 29], which is an effective tool for uncertainty management and uncertainty reasoning, and has a wide variety of applications in artificial intelligence [6, 11, 12, 38, 41, 56]. In the rough set theory, an attribute set partitions a universe into some knowledge granules or elemental concepts, which is called a knowledge structure. Partition, granulation and approximation are the methods widely used in human's reasoning [53-55]. To date, rough set methodology has been applied in feature selection [44, 45], knowledge reduction [16, 21, 26, 39, 47, 51, 57], rule extraction [2, 8, 34, 46, 56, 58, 59], uncertainty reasoning [9, 28, 32] and granular computing [1, 13, 22, 23, 31, 50-52, 54]. In the past ten years, some extensions of Pawlak's rough set model have been proposed in terms of various requirements [5, 25, 37, 42, 43, 49].

Knowledge bases and indiscernibility relations are two basic concepts in the rough set theory and assessing the uncertainty of a knowledge structure in a knowledge base is an important research issue. According to whether or not there missing data (null values), knowledge bases are classified into two categories: complete and incomplete [14-17]. In the rough set theory, there are two main approaches for measuring the uncertainty of a knowledge structure in knowledge bases, which are information entropy [3, 4, 6, 10, 18, 24, 33, 40] and knowledge granulation [7, 19, 20, 33, 48].

For a given knowledge base, one of tasks in data mining and knowledge discovery is to generate new knowledge through using known knowledge. However, in rough set theory, the number of knowledge structures is finite in a given knowledge base, which limits the ability of this knowledge base for approximating an unknown concept. This leads to a task for acquiring more knowledge structures from a given knowledge base. To date, the mechanism that how to generate new knowledge structures based on known knowledge structures in knowledge bases have not been widely researched. Therefore, such a mechanism is desirable and very helpful for rule extraction and knowledge discovery from knowledge bases. In addition, knowledge granulation can be used to characterize the degree of the coarseness of a knowledge structure. The finer the knowledge structure is, the smaller the knowledge granulation is. In recent years, several various forms of knowledge granulations have been given in [19, 20, 33]. From these existing knowledge granulations, we find that they all satisfy several same constraints. In other words, there may exist an uniform description for the existing knowledge granulations. In the view of granular computing, an axiom definition of knowledge granulation may be needed in order to measure the uncertainty of knowledge structures in a knowledge base. It is deserved to point out that when the knowledge granulation (or information entropy) of one knowledge structure is equal to that of the other knowledge structure, these two knowledge structures have the same uncertainty. Nevertheless, it does not mean that these two knowledge structures are equivalent each other. That is to say, information entropy and knowledge granulation cannot characterize the difference between any two knowledge structures in a given knowledge base. In fact, we often need to distinguish two knowledge structures for uncertain data processing in some practical applications.

Based on the above these analyses, main objective of this study has three hands, which are establishing a mathematical framework of granular computing in the context of knowledge bases for acquiring more knowledge structures, constructing an axiom

definition of knowledge granulation and giving a knowledge distance among knowledge structures for characterizing the difference among knowledge structures from a knowledge base, respectively.

The rest of the paper is organized as follows. Some basic concepts in rough set theory are briefly reviewed in Section 2. In Section 3, we establish four operators (\cap , \cup , \wr and $-$) on a knowledge base and investigate their operation properties. Noting that (K, \cap, \cup) is an assignment lattice and (K, \cap, \cup, \wr) is a complemented lattice. In Section 4, an axiom definition of knowledge granulation is constructed, under which several existing forms of knowledge granulations become its special instances. In section 5, to characterize the difference among knowledge structures in a knowledge base, the notion of a knowledge distance is defined and some of its major properties are obtained. Finally, Section 6 concludes this paper with some remarks and discussions.

2. Preliminaries

In this section, we will review several basic concepts in rough set theory and knowledge bases. Throughout this paper, we suppose that the universe U is a finite nonempty set.

Let U be a finite and non-empty set (called a universe) and $R \subseteq U \times U$ an equivalence relation on U , then $K = (U, R)$ is called an knowledge structure (also called an approximation space) [27, 30]. The equivalence relation R partitions the set U into disjoint subsets. This partition of the universe is called a quotient set induced by R , denoted by U/R . It represents a very special type of similarity between elements of the universe. If two elements $x, y \in U (x \neq y)$ belong to the same equivalence class, we say that x and y are indistinguishable under the equivalence relation R , i.e., they are equal in R . We denote the equivalence class including x by $E_R(x)$. For our further development, we denote a knowledge structure induced by U/R on U by $K(R)$. In fact, the knowledge structure is formally defined as $K(R) = \{E_R(x) \mid x \in U\}$. Each equivalence class $E_R(x) (x \in U)$ may be viewed as a knowledge granule consisting of indistinguishable elements [35, 50]. The granulation structure induced by an equivalence relation is a partition of the universe.

We say $K = (U, \mathbf{R})$ is a knowledge base, where U is a finite and non-empty set and \mathbf{R} is a family of equivalence relations. Through using a given knowledge structure, one can construct a rough set of any subset on the universe in the following definition.

Definition 1 [27] *Let $K = (U, \mathbf{R})$ be a knowledge base, X a subset of U and $R \in \mathbf{R}$ an equivalence relation, two sets are defined as*

$$\underline{R}X = \bigcup \{E_R(x) \in U/R \mid E_R(x) \subseteq X\}, \quad (1)$$

$$\overline{R}X = \bigcup \{E_R(x) \in U/R \mid E_R(x) \cap X \neq \emptyset\}, \quad (2)$$

where $\underline{R}X$ and $\overline{R}X$ are called R -lower approximation and R -upper approximation with respect to R , respectively. The order pair $\langle \underline{R}X, \overline{R}X \rangle$ is called a rough set of X with respect to the equivalence relation R .

Let $K = (U, \mathbf{R})$ be a knowledge base, if $R (R \in \mathbf{R})$ is an equivalence relation, then one can get a cover of U by $U/R = \{E_R(x) \mid x \in U\}$, i.e., for $\forall x \in U$, one has that $E_R(x) \neq \emptyset$ and $\bigcup_{x \in U} E_R(x) = U$. Obviously, $\forall x, y \in U (x \neq y)$, if x, y are partitioned

into the same equivalence class, then $E_R(x) = E_R(y)$, otherwise $E_R(x) \cap E_R(y) = \emptyset$. One can define a partial relation \preceq as follows: $P \preceq Q$ ($P, Q \in \mathbf{R}$) if and only if, one has $E_P(x_i) \subseteq E_Q(x_i)$ for any $i \in \{1, 2, \dots, |U|\}$ [20, 36, 50]. Here, we denote that P is finer than Q by $P \preceq Q$. Obviously, (\mathbf{R}, \preceq) is a poset [50].

Another important binary relation is tolerance relation, which satisfies reflexivity and symmetry. For example, in an incomplete information system $S = (U, A)$, we define a binary relation on U by $SIM(A) = \{(x, y) \in U \times U \mid \forall a \in A, a(x) = a(y) \text{ or } a(x) = * \text{ or } a(y) = *\}$, where $*$ is a missing value. Clearly, $SIM(A)$ is a tolerance relation on U . Similarly, let $R \subseteq U \times U$ denote a tolerance relation on U , the tolerance relation R classifies the universe U into some subsets, i.e., a cover of U [14, 15]. This cover of the universe is called a knowledge structure induced by R , denoted by U/R or $K(R)$. If y belongs to the tolerance class determined by x with respect to R , we say two elements x, y are indistinguishable under the tolerance relation R , i.e., they are similar in R [14-17]. We denote the tolerance class of x by $S_R(x)$ [14, 15, 21]. Each tolerance class $S_R(x)$ ($x \in R$) is viewed as a knowledge granule [20, 33, 36]. The granulation structure induced by a tolerance relation is a cover of the universe. Conveniently, we say $K = (U, \mathbf{R})$ is also a knowledge base, where U is a finite and non-empty set and \mathbf{R} is a family of tolerance relations. The following definition gives a rough set of a subset of the universe based on a tolerance relation.

Definition 2 [14, 15] *Let $K = (U, \mathbf{R})$ be a knowledge base, X a subset of U and $R \in \mathbf{R}$ a tolerance relation, two sets are defined as*

$$\underline{R}X = \bigcup \{x \in X \mid S_R(x) \subseteq X\}, \quad (3)$$

$$\overline{R}X = \bigcup \{S_R(x) \mid x \in X\}, \quad (4)$$

where $\underline{R}X$ and $\overline{R}X$ are called R -lower approximation and R -upper approximation with respect to R , respectively. The order pair $\langle \underline{R}X, \overline{R}X \rangle$ is called a rough set of X with respect to the tolerance relation R .

Let $K = (U, \mathbf{R})$ be a knowledge base, if $R(R \in \mathbf{R})$ is a tolerance relation, then we denote a cover of U by $U/R = \{S_R(x) \mid x \in U\}$, i.e., $\forall x \in U$, one has $S_R(x) \neq \emptyset$ and $\bigcup_{x \in U} S_R(x) = U$. In [20], Liang et al. defined a partial relation \preceq as follows: $P \preceq Q$ ($P, Q \in \mathbf{R}$) if and only if, for every $i \in \{1, 2, \dots, |U|\}$, one has that $S_P(x_i) \subseteq S_Q(x_i)$. Here, we also denote that P is finer than Q by $P \preceq Q$. It is easy to see that (\mathbf{R}, \preceq) is also a poset.

3. Operators on a knowledge base

In this section, by uniformly representing a complete knowledge structure and an incomplete knowledge structure, we will propose four operators on a knowledge base and discuss their fundamental algebra properties.

In [20], liang et al. established the relationship between a complete knowledge structure and an incomplete knowledge structure in the same knowledge base. Let $K = (U, R)$ be a knowledge structure, R a equivalence relation, $U/R = \{X_1, X_2, \dots, X_m\}$,

$U/R = \{S_R(x_1), S_R(x_2), \dots, S_R(x_{|U|})\}$ and $X_i = \{x_{i1}, x_{i2}, \dots, x_{is_i}\}$, where $|X_i| = s_i$ and $\sum_{i=1}^m s_i = |U|$, then

$$X_i = S_R(x_{i1}) = S_R(x_{i2}) = \dots = S_R(x_{is_i}). \quad (5)$$

Through this mechanism, one can denote $U/R = \{E_R(x) \mid x \in U\}$ by using $U/R = \{S_R(x) \mid x \in U\}$. The mechanism gives uniform representations of knowledge structures in a knowledge base. It is illustrated by the following example.

Example 1 Let $U = \{x_1, x_2, \dots, x_6\}$, R a equivalence relation and $U/R = \{\{x_1, x_2\}, \{x_3, x_4, x_5\}, \{x_6\}\}$. Then, $U/R = \{S_R(x) \mid x \in U\}$ can be represented equivalently as

$$\begin{aligned} U/R &= \{S_R(x_1), S_R(x_2), S_R(x_3), S_R(x_4), S_R(x_5), S_R(x_6)\} \\ &= \{\{x_1, x_2\}, \{x_1, x_2\}, \{x_3, x_4, x_5\}, \{x_3, x_4, x_5\}, \{x_3, x_4, x_5\}, \{x_6\}\}. \end{aligned}$$

For convenience, we denote the knowledge structure induced by R on U as $K(R)$ in the rest of this paper, where R is an equivalence relation or a tolerance relation.

There are two types of operators to be considered in granular computing based on rough set theory. One is operations among knowledge granules, the other is operations among knowledge structures in a knowledge base. As operations among knowledge granules is based on classical sets, we still operate on them by \cap , \cup , $-$ and \sim , i.e., a new knowledge granule can be generated by \cap , \cup , $-$ and \sim on known knowledge granules. However, operations among knowledge structures are performed through composing and decomposing known knowledge structures in knowledge bases in essence. Therefore, the operators on a knowledge base to generate new knowledge structures are very desirable. In the following, we introduce four operators among knowledge structures in a knowledge base.

Definition 3 Let $K = (U, \mathbf{R})$ be a knowledge base and $K(P), K(Q) \in K$ two knowledge structures. Four operators \cap , \cup , $-$ and \wr on K are defined as

$$K(P) \cap K(Q) = \{S_{P \cap Q}(x) \mid S_{P \cap Q}(x) = S_P(x) \cap S_Q(x), x \in U\}, \quad (6)$$

$$K(P) \cup K(Q) = \{S_{P \cup Q}(x) \mid S_{P \cup Q}(x) = S_P(x) \cup S_Q(x), x \in U\}, \quad (7)$$

$$K(P) - K(Q) = \{S_{P-Q}(x) \mid S_{P-Q}(x) = x \cup (S_P(x) - S_Q(x)), x \in U\}, \quad (8)$$

$$\wr K(P) = \{\wr S_P(x) \mid \wr S_P(x) = x \cup \sim S_P(x), x \in U\}, \quad (9)$$

where $\sim S_P(x) = U - S_P(x)$.

Here, we regard \cap , \cup , $-$ and \wr as four atomic formulas and finite connection on them are all formulas. Through using these operators, one can obtain a new knowledge structure via some known knowledge structures on U . Let $\mathbf{K}(U)$ denote the set of all knowledge structures on U , then these four operators \cap , \cup , $-$ and \wr on $\mathbf{K}(U)$ are close. As follows, we investigate several fundamental algebra properties of these four operators.

Theorem 1 Let \cap , \cup be two operators on K , then

- (1) $K(P) \cap K(P) = K(P)$, $K(P) \cup K(P) = K(P)$;
- (2) $K(P) \cap K(Q) = K(Q) \cap K(P)$, $K(P) \cup K(Q) = K(Q) \cup K(P)$;
- (3) $K(P) \cap (K(P) \cup K(Q)) = K(P)$, $K(P) \cup (K(P) \cap K(Q)) = K(P)$; and

$$(4) \quad (K(P) \cap K(Q)) \cap K(R) = K(P) \cap (K(Q) \cap K(R)), \\ (K(P) \cup K(Q)) \cup K(R) = K(P) \cup (K(Q) \cup K(R)).$$

Proof. They are straightforward from definition 3.

Theorem 2 Let \cap , \cup and \imath be three operators on K , then

- (1) $\imath(\imath K(P)) = K(P)$,
- (2) $K(P) \cap \imath K(P) = \{x_i \mid x_i \in U\}$,
- (3) $\imath(K(P) \cap K(Q)) = \imath K(P) \cup \imath K(Q)$, and
- (4) $\imath(K(P) \cup K(Q)) = \imath K(P) \cap \imath K(Q)$.

Proof. For any $x_i \in U$, $K(P), K(Q) \in K$, $S_P(x_i)$ is the tolerance class induced by x_i in $K(P)$.

(1) From Definition 3, one can easily see that $\imath(S_P(x_i)) = x_i \cup \sim S_P(x_i)$ and $\imath(\imath(S_P(x_i))) = x_i \cup (x_i \cup S_P(x_i)) = S_P(x_i)$. Therefore, $\imath(\imath K(P)) = K(P)$.

(2) From Definition 3, it follows that $S_P(x_i) \cap \imath(S_P(x_i)) = x_i$, $\forall x_i \in U$. Then, $K(P) \cap \sim K(P) = \{x_i \mid x_i \in U\}$.

(3) According to Definition 3, for $\forall x_i \in U$, it follows that

$$\begin{aligned} \imath(S_P(x_i) \cap S_Q(x_i)) &= x_i \cup \sim (S_P(x_i) \cap S_Q(x_i)) \\ &= x_i \cup (\sim S_P(x_i) \cup \sim S_Q(x_i)) \\ &= (x_i \cup \sim S_P(x_i)) \cup (x_i \cup \sim S_Q(x_i)) \\ &= \imath S_P(x_i) \cup \imath S_Q(x_i). \end{aligned}$$

Therefore, one can get that $\imath(K(P) \cap K(Q)) = \imath K(P) \cup \imath K(Q)$.

(4) According to Definition 3, for $\forall x_i \in U$, one has that

$$\begin{aligned} \imath(S_P(x_i) \cup S_Q(x_i)) &= x_i \cup \sim (S_P(x_i) \cup S_Q(x_i)) \\ &= x_i \cup (\sim S_P(x_i) \cap \sim S_Q(x_i)) \\ &= (x_i \cup \sim S_P(x_i)) \cap (x_i \cup \sim S_Q(x_i)) \\ &= \imath S_P(x_i) \cap \imath S_Q(x_i). \end{aligned}$$

Hence, one can obtain that $\imath(K(P) \cup K(Q)) = \imath K(P) \cap \imath K(Q)$. \square

Theorem 2 shows that (1) is reflexive, (2) is complementary, and (3) and (4) are two dual principles.

Theorem 3 Let \cap , \cup , $-$ and \imath be operators on K , then

- (1) $K(P) - K(Q) = K(P) \cap \imath K(Q)$,
- (2) $K(P) - K(Q) = K(P) - (K(P) \cap K(Q))$,
- (3) $K(P) \cap (K(Q) - K(R)) = (K(P) \cap K(Q)) - (K(P) \cap K(R))$, and
- (4) $(K(P) - K(Q)) \cup K(Q) = K(P)$.

Proof. They are straightforward from definition 3.

The above three theorems are illustrated by the following example.

Example 2 Let $U = \{x_1, x_2, x_3, x_4\}$, $K(P) = \{\{x_1, x_2\}, \{x_1, x_2\}, \{x_3, x_4\}, \{x_3, x_4\}\}$ and $K(Q) = \{\{x_1, x_4\}, \{x_2, x_3\}, \{x_2, x_3, x_4\}, \{x_4\}\}$, one can acquire some new knowledge structures through using $K(P)$ and $K(Q)$.

By computing, some new knowledge structures constructed are listed as follows.

$$\begin{aligned} \imath K(P) &= \{\{x_1, x_3, x_4\}, \{x_2, x_3, x_4\}, \{x_1, x_2, x_3\}, \{x_1, x_2, x_4\}\}, \\ \imath K(Q) &= \{\{x_1, x_2, x_3\}, \{x_1, x_2, x_4\}, \{x_1, x_3\}, \{x_1, x_2, x_3, x_4\}\}, \\ K(P) \cap K(Q) &= \{\{x_1\}, \{x_2\}, \{x_3, x_4\}, \{x_4\}\}, \\ K(P) \cup K(Q) &= \{\{x_1, x_2, x_4\}, \{x_1, x_2, x_3\}, \{x_2, x_3, x_4\}, \{x_1, x_4\}\}, \end{aligned}$$

$$\begin{aligned}
\imath K(P) \cap \imath K(Q) &= \{\{x_1, x_3\}, \{x_2, x_4\}, \{x_1, x_3\}, \{x_1, x_2, x_4\}\}, \\
\imath K(P) \cup \imath K(Q) &= \{\{x_1, x_2, x_3, x_4\}, \{x_1, x_2, x_3, x_4\}, \{x_1, x_2, x_3\}, \{x_1, x_2, x_3, x_4\}\}, \\
K(P) - K(Q) &= \{\{x_1, x_2\}, \{x_1, x_2\}, \{x_3\}, \{x_3, x_4\}\} \text{ and} \\
\imath K(P) - \imath K(Q) &= \{\{x_1, x_4\}, \{x_2, x_3\}, \{x_2, x_3\}, \{x_4\}\}. \quad \square
\end{aligned}$$

Suppose $K = (U, R)$ be a knowledge base, $P, Q \in R$, and $K(P), K(Q) \in K$ be two knowledge structures induced by P, Q , respectively. To investigate properties of the operations among knowledge structures on a knowledge base, we will denote $K(P) \preceq K(Q)$ iff $P \preceq Q$.

Theorem 4 Let \cap, \cup and \imath be three operators on K , the following properties hold:

- (1) If $K(P) \preceq K(Q)$, then $\imath K(Q) \preceq \imath K(P)$;
- (2) $K(P) \cap K(Q) \preceq K(P)$, $K(P) \cap K(Q) \preceq K(Q)$; and
- (3) $K(P) \preceq K(P) \cup K(Q)$, $K(Q) \preceq K(P) \cup K(Q)$.

Proof. The terms (2) and (3) can be easily proved from (6) and (7) in Definition 3, respectively.

From Definition 3, one can obtain that

$$\begin{aligned}
K(P) \preceq K(Q) &\implies \text{for } \forall x_i \in U, S_P(x_i) \subseteq S_Q(x_i) \\
&\implies \text{for } \forall x_i \in U, \sim S_Q(x_i) \subseteq \sim S_P(x_i) \\
&\implies \text{for } \forall x_i \in U, x_i \cup \sim S_Q(x_i) \subseteq x_i \cup \sim S_P(x_i) \\
&\implies \imath K(Q) \preceq \imath K(P).
\end{aligned}$$

Hence, the term (1) in this theorem holds. \square

Definition 4 [57] Let (L, \leq) be a poset, if there exist two operators \wedge, \vee on $L : L^2 \longrightarrow L$ such that

- (1) $a \wedge b = b \wedge a$, $a \vee b = b \vee a$,
- (2) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$, $(a \vee b) \vee c = a \vee (b \vee c)$, and
- (3) $a \wedge b = b \iff b \leq a$, $a \vee b = b \iff a \leq b$.

Then we call L is a lattice.

Furthermore, if

- (4) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

Then we call L is an assignment lattice.

We call L a complemented lattice, if for any $a \in L$, there exists a' such that $(a')' = a$ and $a \leq b \iff b' \leq a'$. If there exist $0, 1 \in L$ such that $0 \leq a \leq 1$ for any $a \in L$, then we call 0 and 1 its minimal element and maximal element, respectively.

Theorem 5 (K, \cup, \cap) is an assignment lattice.

Proof. At first, we prove (K, \preceq) is a lattice.

From (2) and (4) in Theorem 1, the terms (1) and (2) in Definition 4 are obvious.

Let $K(P), K(Q), K(R) \in K$ be three knowledge structures, where $K(P) = \{S_P(x) \mid x \in U\}$, $K(Q) = \{S_Q(x) \mid x \in U\}$ and $K(R) = \{S_R(x) \mid x \in U\}$. one can obtain that

$$\begin{aligned}
K(P) \cap K(Q) = K(P) &\iff \text{for } \forall x_i \in U, S_{P \cap Q}(x_i) = S_P(x_i), i \leq |U| \\
&\iff S_P(x_i) \cap S_Q(x_i) = S_P(x_i) \\
&\iff S_P(x_i) \subseteq S_Q(x_i), \text{ for } \forall x_i \in U \\
&\iff K(P) \preceq K(Q).
\end{aligned}$$

According to the dual principle a in lattice, one can easily get that $K(P) \cup K(Q) = K(P) \iff K(Q) \preceq K(P)$. Thus, the term (3) in definition 4 holds.

In addition, for $K(P), K(Q), K(R) \in K$, we know that

$$S_P(x_i) \cap (S_Q(x_i) \cup S_R(x_i)) = (S_P(x_i) \cap (S_Q(x_i))) \cup (S_P(x_i) \cap S_R(x_i)), \forall x_i (i \leq |U|).$$

Hence, $K(P) \cap (K(Q) \cup K(R)) = (K(P) \cap K(Q)) \cup (K(P) \cap K(R))$. From the dual principle in a lattice, one can get that

$$K(P) \cup (K(Q) \cap K(R)) = (K(P) \cup K(Q)) \cap (K(P) \cup K(R)).$$

Therefore, (K, \cup, \cap) is an assignment lattice. \square

Theorem 6 Let $\mathbf{K}(U)$ be the set of all knowledge structures on U , then $(\mathbf{K}(U), \cup, \cap, \imath)$ is a complemented lattice.

Proof. From Theorem 5, it is obvious that $(\mathbf{K}(U), \cup, \cap, \imath)$ is an assignment lattice. Furthermore, from (1) in Theorem 2, one can get that $\imath(\imath K(P)) = K(P)$. In addition, from (3) in Definition 3, one has that

$$\begin{aligned} K(P) \preceq K(Q) &\iff \text{for } \forall x_i \in U, S_P(x_i) \subseteq S_Q(x_i) \\ &\iff \text{for } \forall x_i \in U, \sim S_P(x_i) \supseteq \sim S_Q(x_i) \\ &\iff \text{for } \forall x_i \in U, x_i \cup \sim S_P(x_i) \supseteq x_i \cup \sim S_Q(x_i) \\ &\iff \text{for } \forall x_i \in U, \imath S_P(x_i) \supseteq \imath S_Q(x_i) \\ &\iff K(Q) \preceq K(P). \end{aligned}$$

Hence, $(\mathbf{K}(U), \cup, \cap, \imath)$ is a complemented lattice. \square

In a complemented lattice $(\mathbf{K}(U), \cup, \cap, \imath)$, the knowledge structure $K(\omega) = \{x_i \mid x_i \in U\}$ and the knowledge structure $K(\delta) = \{S_P(x_i) \mid S_P(x_i) = U, x_i \in U\}$ are two special knowledge structures, where $K(\omega)$ is the discrete classification and $K(\delta)$ is the indiscrete classification. For any $K(P) \in \mathbf{K}(U)$, one has that $K(\omega) \preceq K(P) \preceq K(\delta)$. Then, we can call $K(\omega)$ and $K(\delta)$ the minimal element and the maximal element on the lattice $(\mathbf{K}(U), \cup, \cap, \imath)$, respectively.

Remark. One of the strengths of rough set theory is the fact that an unknown target concept can be characterized approximately by existing knowledge structures in a knowledge base. From the above analyses, it is shown that these four operators (\cup , \cap , \imath and $-$) can be applied to generate new knowledge structures on a knowledge base. That is to say, one can use these new knowledge structures to approximate an unknown target. Therefore, this mechanism may be used to rule extraction and knowledge discovery from knowledge bases.

4. Knowledge granulation

As we know, knowledge granulation, in a broad sense, is the average measure of knowledge granules of a knowledge structure in a given knowledge base. It can be used to characterize the classification ability of a given knowledge structure [19, 20, 33, 50, 55].

In recent years, some researchers have discussed and investigated that how to measure the classification ability of a knowledge structure and what is the essence of knowledge granulation in knowledge bases. Generally, the partial relation “ \preceq ” are concerned for investigating various definitions of knowledge granulation. However, the partial relation “ \preceq ” may be not strict in terms of characterizing the properties of knowledge granulation in knowledge bases. In order to discover the essence of knowledge granulation, we introduce a new binary relation “ \ll ” on $\mathbf{K}(U)$ in the following.

Let $K = (U, \mathbf{R})$ be a knowledge base, $P, Q \in \mathbf{R}$, $K(P) = \{S_P(x) \mid x \in U\}$ and

$K(Q) = \{S_Q(x) \mid x \in U\}$. We define a binary relation \leq as follows: $K(P) \leq K(Q)$ ($P, Q \in \mathbf{R}$) if and only if, for $K(P) = \{S_P(x_1), S_P(x_2), \dots, S_P(x_{|U|})\}$, there exists a sequence $K'(Q)$ of $K(Q)$, where $K'(Q) = \{S_Q(x'_1), S_Q(x'_2), \dots, S_Q(x'_{|U|})\}$, such that $|S_P(x_i)| \leq |S_Q(x'_i)|$, $i \leq |U|$.

The following theorem gives some properties of the binary relation \leq .

Theorem 7 *The following properties hold*

- (1) \leq is reflexive,
- (2) \leq is transitive,
- (3) $K(P) \cap K(Q) \leq K(P)$, $K(P) \cap K(Q) \leq K(Q)$, and
- (4) $K(P) \leq K(P) \cup K(Q)$, $K(Q) \leq K(P) \cup K(Q)$.

Proof. They are straightforward.

Here, we say that $K(P)$ is granulation finer than $K(Q)$ if $K(P) \leq K(Q)$. If $K(P) \leq K(Q)$ and there exists $S_Q(x'_i) \in K'(Q)$ such that $|S_P(x_i)| < |S_Q(x'_i)|$, we say that $K(Q)$ is strictly granulation coarser than $K(P)$ (or $K(P)$ is strictly granulation finer than $K(Q)$), denoted by $K(P) \ll K(Q)$.

Theorem 8 *The partial relation \preceq is a special instance of the relation \leq .*

Proof. Let $K = (U, \mathbf{R})$ be a knowledge base, $P, Q \in \mathbf{R}$, $K(P) = \{S_P(x) \mid x \in U\}$ and $K(Q) = \{S_Q(x) \mid x \in U\}$. If $K(P) \preceq K(Q)$, one can obtain that $S_P(x_i) \subseteq S_Q(x_i)$ for any $x_i \in U$, i.e., $|S_P(x_i)| \leq |S_Q(x_i)|$. That is to say, one can find an array of all tolerance classes in $K(Q)$ such that $K(P) \leq K(Q)$. Therefore, the partial relation \preceq is a special instance of the relation \leq . \square

Definition 5 *Let $K = (U, \mathbf{R})$ be a knowledge base, if for $\forall P \in \mathbf{R}$, there is a real number $G(P)$ with the following properties:*

- (1) $G(P) \geq 0$; (Non-negative)
 - (2) for $\forall P, Q \in \mathbf{R}$, let $K(P) = \{S_P(x_i) \mid x_i \in U\}$ and $K(Q) = \{S_Q(x_i) \mid x_i \in U\}$, if there is a bijective mapping function $f : K(P) \rightarrow K(Q)$ such that $|S_P(x_i)| = |f(S_P(x_i))|$, then $G(P) = G(Q)$; (Invariability)
 - (3) if $\forall P, Q \in \mathbf{R}$ and $K(P) \ll K(Q)$, then $G(P) < G(Q)$. (Monotonicity)
- Then G is called a knowledge granulation on K .

As a result of the above discussions, we come to the following three theorems.

Theorem 9 (Extremum) *Let $K = (U, \mathbf{R})$ be a knowledge base and $\forall K(P) \in K$, then $G(P)$ achieves its minimum value if $U/P = \omega$ and $G(P)$ achieves its maximum value if $U/P = \delta$, where ω denotes the identity relation and δ denotes the universal relation.*

Proof. Let $K(\omega) = \{S_\omega(x_i) \mid S_\omega(x_i) = x_i, x_i \in U\}$ and $K(\delta) = \{S_\delta(x_i) \mid S_\delta(x_i) = U, x_i \in U\}$. Hence, for $\forall R \in \mathbf{R}$, $K(R) = \{S_R(x_i) \mid x_i \in U\}$, one has that $x_i \subseteq S_R(x_i)$ and $1 \leq |S_R(x_i)|$, i.e., $K(\omega) \leq K(R)$, and $S_R(x_i) \subseteq U$ and $|S_R(x_i)| \leq |U|$, i.e., $K(R) \leq K(\delta)$. Therefore, $K(\omega) \ll K(R) \ll K(\delta)$, $\forall R \in \mathbf{R}$. From (3) in Definition 5, one can get that $G(\omega) \leq G(R) \leq G(\delta)$, i.e., $G(P)$ achieves its minimum value if $U/P = \omega$ (identity relation) and $G(P)$ achieves its maximum value if $U/P = \delta$ (universal relation). \square

From Definition 5 and Theorem 9, it is easy to see that the size of $G(P)$ only depends on the cardinality of every class in the knowledge structure $K(P)$. The minimum value of $G(P)$ can be obtained when $P = \omega$ and the maximum value of $G(P)$ can be approached when $P = \delta$.

Theorem 10 *Let $K = (U, \mathbf{R})$ be a knowledge base and $K(P), K(Q)$ two knowledge structures on K , then $G(P) \leq G(Q)$ if $K(P) \preceq K(Q)$.*

Proof. From the definition of \preceq , one can see that $K(P) \preceq K(Q)$ ($P, Q \in \mathbf{R}$) if and only if $S_P(x_i) \subseteq S_Q(x_i)$ for every $i \in \{1, 2, \dots, |U|\}$. Hence, for every $S_P(x_i) \in K(P)$, there exists $S_Q(x_i) \in K(Q)$ such that $|S_P(x_i)| \leq |S_Q(x_i)|$, i.e., $K(P) \ll K(Q)$. Therefore, one can easily obtain that $G(P) \leq G(Q)$ from (3) in Definition 5. \square

Theorem 11 *The following properties hold:*

- (1) $G(P) = G(\lambda P)$;
- (2) $G(P \cap Q) \leq G(P)$, $G(P \cap Q) \leq G(Q)$;
- (3) $G(P) \leq G(P \cup Q)$, $G(Q) \leq G(P \cup Q)$; and
- (4) $G(P \cap \lambda P) = G(\omega)$, $G(P \cup \lambda P) = G(\delta)$.

Proof. They are straightforward.

In [19, 20, 33], several different kinds of knowledge granulations have been given, in the following, we prove that these knowledge granulations are all special forms under Definition 6.

Definition 6 [19] *Let $K = (U, \mathbf{R})$ be a knowledge base, $R \in \mathbf{R}$ and $U/R = \{X_1, X_2, \dots, X_m\}$. Knowledge granulation of the knowledge structure $K(R)$ is defined as*

$$GK(R) = \frac{1}{|U|^2} \sum_{i=1}^m |X_i|^2, \frac{1}{|U|} \leq GK(R) \leq 1, \quad (10)$$

where $\sum_{i=1}^m |X_i|^2$ is the cardinality of the equivalence relation $\bigcup_{i=1}^m (X_i \times X_i)$ determined by R .

Theorem 12 *GK in Definition 6 is a knowledge granulation under Definition 5.*

Proof. (1) Obviously, it is non-negative.

(2) Let $P, Q \in \mathbf{R}$, $K(P) = \{S_P(x_1), S_P(x_2), \dots, S_P(x_{|U|})\}$ and $K(Q) = \{S_Q(x_1), S_Q(x_2), \dots, S_Q(x_{|U|})\}$. Supposing that there be a bijective mapping function $f: K(P) \rightarrow K(Q)$ such that $|S_P(x_i)| = |f(S_Q(x_i))|$ and $f(S_P(x_i)) = S_Q(x_{j_i})$, $j_i \in \{1, 2, \dots, |U|\}$, then one has that

$$\begin{aligned} GK(P) &= \frac{1}{|U|^2} \sum_{i=1}^m |X_i|^2 = \frac{1}{|U|^2} \sum_{i=1}^{|U|} |S_P(x_i)| \\ &= \frac{1}{|U|^2} \sum_{i=1}^{|U|} |S_Q(x_{j_i})| = \frac{1}{|U|^2} \sum_{i=1}^{|U|} |S_Q(x_i)| \\ &= GK(Q). \end{aligned}$$

(3) Let $P, Q \in \mathbf{R}$ with $K(P) \ll K(Q)$, $K(P) = \{S_P(x_1), S_P(x_2), \dots, S_P(x_{|U|})\}$ and $K(Q) = \{S_Q(x_1), S_Q(x_2), \dots, S_Q(x_{|U|})\}$, then there exists a sequence $K'(Q)$ of $K(Q)$, where $K'(Q) = \{S_Q(x'_1), S_Q(x'_2), \dots, S_Q(x'_{|U|})\}$, such that $|S_P(x_i)| < |S_Q(x'_i)|$. Hence,

one has that

$$\begin{aligned} GK(P) &= \frac{1}{|U|^2} \sum_{i=1}^m |X_i|^2 = \frac{1}{|U|^2} \sum_{i=1}^{|U|} |S_P(x_i)| \\ &< \frac{1}{|U|^2} \sum_{i=1}^{|U|} |S_Q(x'_i)| = \frac{1}{|U|^2} \sum_{i=1}^{|U|} |S_Q(x_i)| \\ &= GK(Q). \end{aligned}$$

Therefore, GK in Definition 6 is a knowledge granulation under Definition 5. \square

Definition 7 [20] Let $K = (U, \mathbf{R})$ be a knowledge base, $P \in \mathbf{R}$ and $K(P) = \{S_P(x_1), S_P(x_2), \dots, S_P(x_{|U|})\}$. Knowledge granulation of the knowledge structure $K(P)$ is defined as

$$G(P) = \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|S_P(x_i)|}{|U|}, \quad (11)$$

where $\frac{|S_P(x_i)|}{|U|}$ is the probability of tolerance class $S_P(x_i)$ within the universe U .

If $K(P) = K(\omega)$, $G(P)$ achieves its minimum value $G(P) = \frac{1}{|U|}$; if $K(P) = K(\delta)$, $G(P)$ achieves its maximum value $G(P) = 1$. It is obvious that $\frac{1}{|U|} \leq G(P) \leq 1$.

Theorem 13 G in Definition 7 is a knowledge granulation under Definition 5.

Proof. (1) Obviously, it is non-negative.

(2) Let $P, Q \in \mathbf{R}$, $K(P) = \{S_P(x_1), S_P(x_2), \dots, S_P(x_{|U|})\}$ and $K(Q) = \{S_Q(x_1), S_Q(x_2), \dots, S_Q(x_{|U|})\}$. Supposing that there be a bijective mapping function $f: K(P) \rightarrow K(Q)$ such that $|S_P(x_i)| = |f(S_Q(x_i))|$ and $f(S_P(x_i)) = S_Q(x_{j_i})$, $j_i \in \{1, 2, \dots, |U|\}$, then one has that

$$\begin{aligned} G(P) &= \frac{1}{|U|^2} \sum_{i=1}^{|U|} |S_P(x_i)| \\ &= \frac{1}{|U|^2} \sum_{i=1}^{|U|} |S_Q(x_{j_i})| = \frac{1}{|U|^2} \sum_{i=1}^{|U|} |S_Q(x_i)| \\ &= G(Q). \end{aligned}$$

(3) Let $P, Q \in \mathbf{R}$ with $K(P) \ll K(Q)$, $K(P) = \{S_P(x_1), S_P(x_2), \dots, S_P(x_{|U|})\}$ and $K(Q) = \{S_Q(x_1), S_Q(x_2), \dots, S_Q(x_{|U|})\}$, then there exists a sequence $K'(Q)$ of $K(Q)$, where $K'(Q) = \{S_Q(x'_1), S_Q(x'_2), \dots, S_Q(x'_{|U|})\}$, such that $|S_P(x_i)| < |S_Q(x'_i)|$. Hence, one has that

$$\begin{aligned} G(P) &= \frac{1}{|U|^2} \sum_{i=1}^{|U|} |S_P(x_i)| \\ &< \frac{1}{|U|^2} \sum_{i=1}^{|U|} |S_Q(x'_i)| = \frac{1}{|U|^2} \sum_{i=1}^{|U|} |S_Q(x_i)| \\ &= G(Q). \end{aligned}$$

Thus, G in Definition 7 is a knowledge granulation under Definition 5. \square

Theorem 14 Let $K = (U, \mathbf{R})$ be a knowledge base, $P \in \mathbf{R}$, $K(P) = \{S_P(x_1), S_P(x_2), \dots, S_P(x_{|U|})\}$ and λP the relation induced by $\lambda K(P)$, then $G(P) + G(\lambda P) = 1 + \frac{1}{|U|}$.

Proof. From Definition 7, it follows that

$$\begin{aligned}
G(P) + G(\lambda P) &= \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|S_P(x_i)|}{|U|} + \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|\lambda S_P(x_i)|}{|U|} \\
&= \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|S_P(x_i)|}{|U|} + \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|x_i \cup \lambda S_P(x_i)|}{|U|} \\
&= \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|U|+1}{|U|} \\
&= 1 + \frac{1}{|U|}.
\end{aligned}$$

That is $G(P) + G(\lambda P) = 1 + \frac{1}{|U|}$. \square

Theorem 15 Let $\mathbf{K}(U)$ be the set of all knowledge structures on U and $K(P), K(Q) \in \mathbf{K}(U)$ two knowledge structures, then $G(P) - G(Q) = G(\lambda(Q)) - G(\lambda(P))$.

Proof. Obviously, we have that

$$\begin{aligned}
G(\lambda P) - G(\lambda Q) &= \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{1+|U|-|S_P(x_i)|}{|U|} - \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{1+|U|-|S_Q(x_i)|}{|U|} \\
&= \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|S_Q(x_i)| - |S_P(x_i)|}{|U|} \\
&= -(G(P) - G(Q)),
\end{aligned}$$

i.e., $G(P) - G(Q) = G(\lambda(Q)) - G(\lambda(P))$. \square

Definition 8 [33] Let $K = (U, \mathbf{R})$ be a knowledge base, $R \in \mathbf{R}$ and $U/R = \{X_1, X_2, \dots, X_m\}$. Combination granulation of knowledge structure $K(R)$ is defined as

$$CG(R) = \sum_{i=1}^m \frac{|X_i|}{|U|} \frac{C_{|X_i|}^2}{C_{|U|}^2}, \quad (12)$$

where $0 \leq CG(R) \leq 1$, $\frac{|X_i|}{|U|}$ represents the probability of equivalence class X_i within the universe U , and $\frac{C_{|X_i|}^2}{C_{|U|}^2}$ denotes the probability of pairs of elements on equivalence class X_i within the whole pairs of elements on the universe U .

Theorem 16 CG in Definition 8 is a knowledge granulation under Definition 5.

Proof. (1) Obviously, it is non-negative.

(2) Let $P, Q \in \mathbf{R}$, $K(P) = \{S_P(x_1), S_P(x_2), \dots, S_P(x_{|U|})\}$ and $K(Q) = \{S_Q(x_1), S_Q(x_2), \dots, S_Q(x_{|U|})\}$. Supposing that there be a bijective mapping function $f: K(P) \rightarrow K(Q)$ such that $|S_P(x_i)| = |f(S_Q(x_i))|$ and $f(S_P(x_i)) = S_Q(x_{j_i})$, $j_i \in \{1, 2, \dots, |U|\}$. Then,

$$\begin{aligned}
CG(P) &= \sum_{i=1}^m \frac{|P_i|}{|U|} \frac{C_{|P_i|}^2}{C_{|U|}^2} = \sum_{i=1}^{|U|} \frac{|S_P(u_i)|}{|U|} \frac{C_{|S_P(u_i)|}^2}{C_{|U|}^2} \\
&= \sum_{i=1}^{|U|} \frac{|S_Q(u_{j_i})|}{|U|} \frac{C_{|S_Q(u_{j_i})|}^2}{C_{|U|}^2} = \sum_{i=1}^{|U|} \frac{|S_Q(u_i)|}{|U|} \frac{C_{|S_Q(u_i)|}^2}{C_{|U|}^2} \\
&= \sum_{j=1}^n \frac{|Q_j|}{|U|} \frac{C_{|Q_j|}^2}{C_{|U|}^2} = CG(Q).
\end{aligned}$$

(3) Let $P, Q \in \mathbf{R}$ with $P \ll Q$, $K(P) = \{S_P(u_1), S_P(u_2), \dots, S_P(u_{|U|})\}$ and $K(Q) = \{S_Q(u_1), S_Q(u_2), \dots, S_Q(u_{|U|})\}$, then for arbitrary $S_P(u_i) (i \leq |U|)$, there exists a

sequence $\{S_Q(u'_1), S_Q(u'_2), \dots, S_Q(u'_{|U|})\}$ such that $|S_P(u_i)| < |S_Q(u'_i)| (i = 1, 2, \dots, |U|)$. Therefore,

$$\begin{aligned} CG(P) &= \sum_{i=1}^m \frac{|P_i|}{|U|} \frac{C_{|P_i|}^2}{C_{|U|}^2} = \sum_{i=1}^{|U|} \frac{|S_P(u_i)|}{|U|} \frac{C_{|S_P(u_i)|}^2}{C_{|U|}^2} \\ &< \sum_{i=1}^{|U|} \frac{|S_Q(u'_i)|}{|U|} \frac{C_{|S_Q(u'_i)|}^2}{C_{|U|}^2} = \sum_{i=1}^{|U|} \frac{|S_Q(u_i)|}{|U|} \frac{C_{|S_Q(u_i)|}^2}{C_{|U|}^2} \\ &= \sum_{j=1}^n \frac{|Q_j|}{|U|} \frac{C_{|Q_j|}^2}{C_{|U|}^2} = CG(Q). \end{aligned}$$

Hence, CG in Definition 8 is a knowledge granulation under Definition 5. \square

Definition 9 [33] Let $K = (U, \mathbf{R})$ be a knowledge base, $P \in \mathbf{R}$, $K(P) = \{S_P(x_1), S_P(x_2), \dots, S_P(x_{|U|})\}$. Combination granulation of knowledge structure $K(P)$ is defined by

$$CG(P) = \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{C_{|S_P(x_i)|}^2}{C_{|U|}^2}. \quad (13)$$

Where $\frac{C_{|S_P(x_i)|}^2}{C_{|U|}^2}$ denotes the probability of pairs of elements on tolerance class $S_P(x_i)$ within the whole pairs of elements on the universe U .

Theorem 17 CG in Definition 9 is a knowledge granulation under Definition 5.

proof. (1) Obviously, it is non-negative.

(2) Let $P, Q \in \mathbf{R}$, $K(P) = \{S_P(x_1), S_P(x_2), \dots, S_P(x_{|U|})\}$ and $K(Q) = \{S_Q(x_1), S_Q(x_2), \dots, S_Q(x_{|U|})\}$. Supposing that there be a bijective mapping function $f : K(P) \rightarrow K(Q)$ such that $|S_P(x_i)| = |f(S_Q(x_i))|$ and $f(S_P(x_i)) = S_Q(x_{j_i})$, $j_i \in \{1, 2, \dots, |U|\}$, one has that

$$\begin{aligned} CG(P) &= \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{C_{|S_P(x_i)|}^2}{C_{|U|}^2} = \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{C_{|S_Q(x_{j_i})|}^2}{C_{|U|}^2} \\ &= \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{C_{|S_Q(x_i)|}^2}{C_{|U|}^2} \\ &= CG(Q). \end{aligned}$$

(3) Let $P, Q \in \mathbf{R}$ with $P \ll Q$, $K(P) = \{S_P(u_1), S_P(u_2), \dots, S_P(u_{|U|})\}$ and $K(Q) = \{S_Q(u_1), S_Q(u_2), \dots, S_Q(u_{|U|})\}$, then for any $S_P(u_i) (i \leq |U|)$, there exists a sequence $\{S_Q(u'_1), S_Q(u'_2), \dots, S_Q(u'_{|U|})\}$ such that $|S_P(u_i)| < |S_Q(u'_i)| (i = 1, 2, \dots, |U|)$. Therefore,

$$\begin{aligned} CG(P) &= \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{C_{|S_P(u_i)|}^2}{C_{|U|}^2} \\ &< \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{C_{|S_Q(u'_i)|}^2}{C_{|U|}^2} = \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{C_{|S_Q(u_i)|}^2}{C_{|U|}^2} \\ &= CG(Q). \end{aligned}$$

Thus, CG in Definition 9 is a knowledge granulation under Definition 5. \square

Through using the axiom definition of knowledge granulation, one can construct some new knowledge granulations according to various opinions. In the following, we show the significance of the axiom definition of knowledge granulation by constructing a new form

of knowledge granulation in a given knowledge base.

Definition 10 Let $K = (U, \mathbf{R})$ be a knowledge base, $P \in \mathbf{R}$ and $K(P) = \{S_P(x_1), S_P(x_2), \dots, S_P(x_{|U|})\}$. Knowledge granulation of the knowledge structure $K(P)$ is defined as

$$GE(P) = \frac{1}{|U|} \sum_{i=1}^{|U|} \log_2 |S_P(x_i)|. \quad (14)$$

If $K(P) = K(\omega)$, $G(P)$ achieves its minimum value $G(P) = 0$; if $K(P) = K(\delta)$, $G(P)$ achieves its maximum value $G(P) = \log_2 |U|$. It is obvious that $0 \leq GE(P) \leq \log_2 |U|$.

Theorem 18 GE in Definition 10 is a knowledge granulation under Definition 5.

Proof. (1) Obviously, it is non-negative.

(2) Let $P, Q \in \mathbf{R}$, $K(P) = \{S_P(x_1), S_P(x_2), \dots, S_P(x_{|U|})\}$ and $K(Q) = \{S_Q(x_1), S_Q(x_2), \dots, S_Q(x_{|U|})\}$. Supposing that there be a bijective mapping function $f: K(P) \rightarrow K(Q)$ such that $|S_P(x_i)| = |f(S_Q(x_i))|$ and $f(S_P(x_i)) = S_Q(x_{j_i})$, $j_i \in \{1, 2, \dots, |U|\}$. Hence,

$$\begin{aligned} GE(P) &= \frac{1}{|U|} \sum_{i=1}^{|U|} \log_2 |S_P(x_i)| \\ &= \frac{1}{|U|} \sum_{i=1}^{|U|} \log_2 |S_Q(x_{j_i})| = \frac{1}{|U|} \sum_{i=1}^{|U|} \log_2 |S_Q(x_i)| \\ &= GE(Q). \end{aligned}$$

(3) Let $P, Q \in \mathbf{R}$ with $K(P) \ll K(Q)$, $K(P) = \{S_P(x_1), S_P(x_2), \dots, S_P(x_{|U|})\}$ and $K(Q) = \{S_Q(x_1), S_Q(x_2), \dots, S_Q(x_{|U|})\}$, then there exists a sequence $K'(Q)$ of $K(Q)$, where $K'(Q) = \{S_Q(x'_1), S_Q(x'_2), \dots, S_Q(x'_{|U|})\}$, such that $|S_P(x_i)| < |S_Q(x'_i)|$. Hence,

$$\begin{aligned} GE(P) &= \frac{1}{|U|} \sum_{i=1}^{|U|} \log_2 |S_P(x_i)| \\ &\leq \frac{1}{|U|} \sum_{i=1}^{|U|} \log_2 |S_Q(x'_i)| = \frac{1}{|U|} \sum_{i=1}^{|U|} \log_2 |S_Q(x_i)| \\ &= GE(Q). \end{aligned}$$

Therefore, GE in Definition 10 is a knowledge granulation under Definition 5. \square

5. Knowledge distance

In rough set theory, information entropy and knowledge granulation are two main approaches to measuring the uncertainty of a knowledge structure in knowledge bases. If the knowledge granulation (or information entropy) of one knowledge structure is equal to that of the other knowledge structure, we say that these two knowledge structures have the same uncertainty. However, it does not mean that these two knowledge structures are equivalent each other. In other words, information entropy and knowledge granulation cannot characterize the difference between any two knowledge structures in a knowledge base. In this section, we introduce a notion of knowledge distance to differentiate two given knowledge structures and investigate some of its important properties.

In [50], Yao presented the concept of set closeness between two classical sets to measure the degree of the sameness between sets. Let A and B be two finite sets, the measure is defined by $H(A, B) = \frac{|A \cap B|}{|A \cup B|}$ ($A \cup B \neq \emptyset$) [50]. Obviously, the formula $1 - H(A, B) = 1 - \frac{|A \cap B|}{|A \cup B|}$ ($A \cup B \neq \emptyset$) can characterize the difference between two finite classical sets. To characterize the relationship among knowledge structures, based on the view of set closeness, we introduce an approach called knowledge distance for measuring the difference between two knowledge structures on the same knowledge base in the following.

Definition 11 Let $K = (U, \mathbf{R})$ be a knowledge base, $P, Q \in \mathbf{R}$, $K(P) = \{S_P(x_i) \mid x_i \in U\}$ and $K(Q) = \{S_Q(x_i) \mid x_i \in U\}$. Knowledge distance between $K(P)$ and $K(Q)$ is defined as

$$D(K(P), K(Q)) = \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|S_P(x_i) \oplus S_Q(x_i)|}{|U|}, \quad (15)$$

where $|S_P(x_i) \oplus S_Q(x_i)| = |S_P(x_i) \cup S_Q(x_i)| - |S_P(x_i) \cap S_Q(x_i)|$.

The knowledge distance represents the measure of difference between two knowledge structures in the same knowledge base. Obviously, $0 \leq D(K(P), K(Q)) \leq 1 - \frac{1}{|U|}$.

Theorem 19 (Extremum) Let $K(U, \mathbf{R})$ be a knowledge base, $K(P), K(Q)$ two knowledge structures on K . Then, $D(K(P), K(Q))$ achieves its minimum value $D(K(P), K(Q)) = 0$ iff $K(P) = K(Q)$ and $D(K(P), K(Q))$ achieves its maximum value $D(K(P), K(Q)) = 1 - \frac{1}{|U|}$ iff $K(P) = \imath K(Q)$ ($\Leftrightarrow K(Q) = \imath K(P)$).

Proof. For $\forall P, Q \in \mathbf{R}$, one has that $1 \leq |S_P(x_i) \cap S_P(x_i)| \leq |U|$ and $1 \leq |S_P(x_i) \cup S_P(x_i)| \leq |U|$. Therefore, $\forall P, Q \in \mathbf{R}$, $0 \leq |S_P(x_i) \oplus S_Q(x_i)| \leq |U| - 1$, i.e., $0 \leq \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|S_P(x_i) \oplus S_Q(x_i)|}{|U|} \leq 1 - \frac{1}{|U|}$.

If $K(P) = K(Q)$, then $K(P) \cap K(Q) = K(P)$ and $K(P) \cup K(Q) = K(P)$. Hence,

$$D(K(P), K(Q)) = \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|S_P(x_i) \oplus S_Q(x_i)|}{|U|} = \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{0}{|U|} = 0,$$

i.e., $D(K(P), K(Q))$ achieves its minimum value 0 if and only if $K(P) = K(Q)$.

If $K(P) = \imath K(Q)$, then $K(P) \cap K(Q) = K(\omega)$ and $K(P) \cup K(Q) = K(\delta)$. Hence,

$$D(K(P), K(Q)) = \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|S_P(x_i) \oplus S_Q(x_i)|}{|U|} = \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|U| - |x_i|}{|U|} = 1 - \frac{1}{|U|},$$

i.e., $D(K(P), K(Q))$ achieves its maximum value $1 - \frac{1}{|U|}$ if and only if $K(P) = \imath K(Q)$ or $K(Q) = \imath K(P)$. \square

In particular, one has that $D(K(\omega), K(\delta)) = 1 - \frac{1}{|U|}$. In fact, we have that $K(\omega) = \imath K(\delta)$ and $K(\delta) = \imath K(\omega)$.

Let $K(P) = \{S_P(x_i) \mid x_i \in U\}$, $K(Q) = \{S_Q(x_i) \mid x_i \in U\}$ and $K(R) = \{S_R(x_i) \mid x_i \in U\}$ be three knowledge structures on U . For $S_P(x_i) \in K(P)$, $S_Q(x_i) \in K(Q)$ and $S_R(x_i) \in K(R)$, $x_i \in U$, we note $S_{P \cup Q \cup R}(x_i) = S_P(x_i) \cup S_Q(x_i) \cup S_R(x_i)$. One can give a certain array of all elements in $S_{P \cup Q \cup R}(x_i)$ and denote the array by $Array = (x_{i_1}, x_{i_2}, \dots, x_{i_{|S_{P \cup Q \cup R}(x_i)|}})$. Therefore, one can represent $S_P(x_i)$ by the following array

$$x_{i_k} = \begin{cases} 1 & \text{if } x_{i_k} \in S_P(x_i), \\ 0 & \text{else,} \end{cases}$$

for $x_{i_k} \in \text{Array}$, $k \leq |S_{P \cup Q \cup R}(x_i)|$.

Similarly, the expressions of $S_Q(x_i)$ and $S_R(x_i)$ can also be obtained. In fact, the expression of Array is various, so the expression of $S_P(x_i)$, $S_Q(x_i)$ and $S_R(x_i)$ should also be changed according to Array , respectively. This kind of representations about the tolerance classes is illustrated by the following example.

Example 3 Consider three tolerance classes $S_P(x_i) = \{1, 2, 3\}$, $S_Q(x_i) = \{2, 3, 4\}$ and $S_R(x_i) = \{3, 4, 5\}$. Compute the expressions of $S_P(x_i)$, $S_Q(x_i)$ and $S_R(x_i)$ through using the above method.

By computing, one has that $S_{P \cup Q \cup R}(x_i) = S_P(x_i) \cup S_Q(x_i) \cup S_R(x_i) = \{1, 2, 3, 4, 5\}$. Assume that $\text{Array} = (1, 2, 3, 4, 5)$.

For $S_P(x_i)$, one can obtain that $S_P(x_i) = (1, 1, 1, 0, 0)$. Similarly, it follows that $S_Q(x_i) = (0, 1, 1, 1, 0)$ and $S_R(x_i) = (0, 0, 1, 1, 1)$.

Let A, B, C be three classical sets, $\text{Array} = (t_1, t_2, \dots, t_{|A \cup B \cup C|})$, $t_i \cap t_j = \emptyset$, $t_i, t_j \in A \cup B \cup C$. Hence, from the above expression method, one can get the array expressions of A, B and C as follows

$$\begin{aligned} A' &= \{a_1, a_2, \dots, a_{|A \cup B \cup C|}\}, \\ B' &= \{b_1, b_2, \dots, b_{|A \cup B \cup C|}\} \text{ and} \\ C' &= \{c_1, c_2, \dots, c_{|A \cup B \cup C|}\}. \end{aligned}$$

Based on these denotations, we then measure the distance between two classical sets by the following formula

$$d(A, B) = \sum_{i=1}^{|A \cup B \cup C|} (a_i + b_i), a_i \in A', b_i \in B'. \quad (16)$$

Analogously, one has that $d(B, C) = \sum_{i=1}^{|A \cup B \cup C|} (b_i + c_i)$ and $d(A, C) = \sum_{i=1}^{|A \cup B \cup C|} (a_i + c_i)$.

From these denotations, we come to the following lemma.

Lemma 1 Let A, B, C be three classical sets, then $d(A, B) + d(B, C) \geq d(A, C)$, $d(A, B) + d(A, C) \geq d(B, C)$ and $d(A, C) + d(B, C) \geq d(A, B)$.

Proof. Suppose that $A' = \{a_1, a_2, \dots, a_{|A \cup B \cup C|}\}$, $B' = \{b_1, b_2, \dots, b_{|A \cup B \cup C|}\}$ and $C' = \{c_1, c_2, \dots, c_{|A \cup B \cup C|}\}$. From $(a_i + b_i) + (b_i + c_i) \geq (a_i + c_i)$, it follows that

$$\begin{aligned} d(A, B) + d(B, C) &= \sum_{i=1}^{|A \cup B \cup C|} (a_i + b_i) + \sum_{i=1}^{|A \cup B \cup C|} (b_i + c_i) \\ &= \sum_{i=1}^{|A \cup B \cup C|} ((a_i + b_i) + (b_i + c_i)) \\ &\geq \sum_{i=1}^{|A \cup B \cup C|} (a_i + c_i) = d(A, C). \end{aligned}$$

Similarly, $d(A, B) + d(A, C) \geq d(B, C)$ and $d(A, C) + d(B, C) \geq d(A, B)$. \square

Theorem 20 Let $\mathbf{K}(U)$ be the set of all knowledge structures induced by U , then $(\mathbf{K}(U), D)$ is a distance space.

Proof. (1) One can obtain easily that $D(K(P), K(Q)) \geq 0$ from Definition 7.

(2) It is obvious that $D(K(P), K(Q)) = D(K(Q), K(P))$.

(3) For the proof of the triangle inequality principle, one only need to prove that $D(K(P), K(Q)) + D(K(P), K(R)) \geq D(K(Q), K(R))$, $D(K(R), K(Q)) + D(K(P), K(R)) \geq D(K(Q), K(P))$ and $D(K(R), K(Q)) + D(K(P), K(Q)) \geq D(K(P), K(R))$ for any $K(P), K(Q), K(R) \in \mathbf{K}(U)$.

From lemma 1, we know that for $x_i \in U$, $D(S_P(x_i), S_Q(x_i)) + D(S_P(x_i), S_R(x_i)) \geq D(S_Q(x_i), S_R(x_i))$, $D(S_P(x_i), S_Q(x_i)) + D(S_Q(x_i), S_R(x_i)) \geq D(S_P(x_i), S_R(x_i))$ and $D(S_P(x_i), S_R(x_i)) + D(S_Q(x_i), S_R(x_i)) \geq D(S_P(x_i), S_Q(x_i))$. Hence,

$$\begin{aligned} & D(K(P), K(Q)) + D(K(P), K(R)) \\ &= \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|S_P(x_i) \oplus S_Q(x_i)|}{|U|} + \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|S_P(x_i) \oplus S_R(x_i)|}{|U|} \\ &= \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{d(S_P(x_i), S_Q(x_i))}{|U|} + \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{d(S_P(x_i), S_R(x_i))}{|U|} \\ &= \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{1}{|U|} (d(S_P(x_i), S_Q(x_i)) + d(S_P(x_i), S_R(x_i))) \\ &\geq \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{d(S_Q(x_i), S_R(x_i))}{|U|} \\ &= \frac{1}{|U|} \sum_{i=1}^{|U|} D(K(Q), K(R)). \end{aligned}$$

Similarly, one can obtain that $D(K(R), K(Q)) + D(K(P), K(R)) \geq D(K(Q), K(P))$, $D(K(R), K(Q)) + D(K(P), K(Q)) \geq D(K(R), K(P))$.

Therefore, $(\mathbf{K}(U), D)$ is a distance space. \square

The above theorem is explained by the following example.

Example 4 Assume that $K = \{x_1, x_2, x_3, x_4, x_5\}$ and $K(P), K(Q), K(R)$ be three knowledge structures induced by equivalence relations P, Q, R on K , where $K(P) = \{\{x_1, x_2, x_3\}, \{x_1, x_2, x_3\}, \{x_1, x_2, x_3\}, \{x_4, x_5\}, \{x_4, x_5\}\}$, $K(Q) = \{\{x_1, x_2\}, \{x_1, x_2\}, \{x_3\}, \{x_4, x_5\}, \{x_4, x_5\}\}$ and $K(R) = \{\{x_1, x_2\}, \{x_1, x_3\}, \{x_3\}, \{x_4\}, \{x_5\}\}$.

By computing their knowledge distances, one can obtain that

$$\begin{aligned} D(K(P), K(Q)) &= \frac{1}{5} \left[\frac{1}{5} (1 + 1 + 2 + 0 + 0) \right] = \frac{4}{25}, \\ D(K(P), K(R)) &= \frac{1}{5} \left[\frac{1}{5} (1 + 1 + 2 + 1 + 1) \right] = \frac{6}{25} \text{ and} \\ D(K(Q), K(R)) &= \frac{1}{5} \left[\frac{1}{5} (0 + 2 + 0 + 1 + 1) \right] = \frac{4}{25}. \end{aligned}$$

Hence, one has that $\frac{4}{25} + \frac{6}{25} = \frac{10}{25} > \frac{4}{25}$, $\frac{4}{25} + \frac{4}{25} = \frac{8}{25} > \frac{6}{25}$. It is easy to see that $D(K(P), K(Q)) + D(K(P), K(R)) \geq D(K(Q), K(R))$, $D(K(P), K(Q)) + D(K(Q), K(R)) \geq D(K(P), K(R))$ and $D(K(Q), K(R)) + D(K(P), K(R)) \geq D(K(P), K(Q))$. \square

For further development, we give the following Lemma 2.

Lemma 2 Let A, B, C be three classical sets with $A \subseteq B \subseteq C$ or $A \supseteq B \supseteq C$, then $d(A, B) + d(B, C) = d(A, C)$.

Proof. Suppose that $A' = \{a_1, a_2, \dots, a_{|A \cup B \cup C|}\}$, $B' = \{b_1, b_2, \dots, b_{|A \cup B \cup C|}\}$ and

$C' = \{c_1, c_2, \dots, c_{|A \cup B \cup C|}\}$. Let $A \subseteq B \subseteq C$, thus $A \cup B \cup C = A$ and $B \cup C = B$. Therefore,

$$\begin{aligned} d(A, B) + d(B, C) &= \sum_{i=1}^{|A \cup B \cup C|} (a_i + b_i) + \sum_{i=1}^{|A \cup B \cup C|} (b_i + c_i) \\ &= (|A \cup B| - |A \cap B|) + (|B \cup C| - |B \cap C|) \\ &= (|A| - |B|) + (|B| - |C|) \\ &= |A| - |C| \\ &= \sum_{i=1}^{|A \cup B \cup C|} (a_i + c_i) \\ &= d(A, C). \end{aligned}$$

For $A \supseteq B \supseteq C$, similarly, one can draw the same conclusion. \square

By Definition 11 and Lemma 2, one can obtain the following theorem.

Theorem 21 *Let $K = (U, \mathbf{R})$ be a knowledge base, $P, Q, R \in \mathbf{R}$ and $K(P) \preceq K(Q) \preceq K(R)$ or $K(R) \preceq K(Q) \preceq K(P)$. Then, $D(K(P), K(R)) = D(K(P), K(Q)) + D(K(Q), K(R))$.*

Proof. For $K(P), K(Q), K(R) \in K$ and $K(P) \preceq K(Q) \preceq K(R)$, one can easily get that $S_P(x_i) \subseteq S_Q(x_i) \subseteq S_R(x_i)$, $x_i \in U$. Hence, it follows from Lemma 2 that

$$\begin{aligned} &D(K(P), K(Q)) + D(K(Q), K(R)) \\ &= \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|S_P(x_i) \oplus S_Q(x_i)|}{|U|} + \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|S_Q(x_i) \oplus S_R(x_i)|}{|U|} \\ &= \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{d(S_P(x_i), S_Q(x_i))}{|U|} + \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{d(S_Q(x_i), S_R(x_i))}{|U|} \\ &= \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{1}{|U|} (d(S_P(x_i), S_Q(x_i)) + d(S_Q(x_i), S_R(x_i))) \\ &= \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{d(S_P(x_i), S_R(x_i))}{|U|} \\ &= \frac{1}{|U|} \sum_{i=1}^{|U|} D(K(P), K(R)). \end{aligned}$$

For $K(R) \preceq K(Q) \preceq K(P)$, similarly, one can draw the same conclusion. \square

Example 5 *Let $U = \{x_1, x_2, x_3, x_4, x_5\}$ and $K(P), K(Q), K(R)$ be three knowledge structures induced by equivalence relations P, Q, R on K , where $K(P) = \{\{x_1, x_2, x_3\}, \{x_1, x_2, x_3\}, \{x_1, x_2, x_3\}, \{x_4, x_5\}, \{x_4, x_5\}\}$, $K(Q) = \{\{x_1, x_2\}, \{x_1, x_2\}, \{x_3\}, \{x_4, x_5\}, \{x_4, x_5\}\}$ and $K(R) = \{\{x_1, x_2\}, \{x_1, x_2\}, \{x_3\}, \{x_4\}, \{x_5\}\}$.*

It is obvious that $K(R) \preceq K(Q) \preceq K(P)$. By computing the knowledge distances among them, one can obtain that

$$\begin{aligned} D(K(P), K(Q)) &= \frac{1}{5} \left[\frac{1}{5} (1 + 1 + 2 + 0 + 0) \right] = \frac{4}{25}, \\ D(K(Q), K(R)) &= \frac{1}{5} \left[\frac{1}{5} (0 + 0 + 0 + 1 + 1) \right] = \frac{2}{25} \text{ and} \\ D(K(P), K(R)) &= \frac{1}{5} \left[\frac{1}{5} (1 + 1 + 2 + 1 + 1) \right] = \frac{6}{25}. \end{aligned}$$

It is clear that $D(K(P), K(Q)) + D(K(Q), K(R)) = \frac{4}{25} + \frac{2}{25} = \frac{6}{25} = D(K(P), K(R))$.

Theorem 22 *Let $\mathbf{K}(U)$ be the set of all knowledge structures induced by U and $K(P), K(Q) \in \mathbf{K}(U)$ two knowledge structures, then $D(K(P), K(Q)) = D(\imath K(P), \imath K(Q))$.*

Proof. It follows from the definition of D that

$$\begin{aligned}
& D(\imath K(P), \imath K(Q)) \\
&= \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|(x_i \cup \sim S_P(x_i)) \oplus (x_i \cup \sim S_Q(x_i))|}{|U|} \\
&= \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|(x_i \cup \sim S_P(x_i)) \cup (x_i \cup \sim S_Q(x_i))| - |(x_i \cup \sim S_P(x_i)) \cup (x_i \cap \sim S_Q(x_i))|}{|U|} \\
&= \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|x_i \cup (\sim S_P(x_i) \cup \sim S_Q(x_i))| - |(x_i \cup (\sim S_P(x_i) \cap \sim S_Q(x_i)))|}{|U|} \\
&= (1 + |U| - |S_P(x_i) \cap S_Q(x_i)|) - (1 + |U| - |S_P(x_i) \cup S_Q(x_i)|) \\
&= |S_P(x_i) \cup S_Q(x_i)| - |S_P(x_i) \cap S_Q(x_i)| \\
&= \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|S_P(x_i) \oplus S_Q(x_i)|}{|U|} \\
&= D(K(P), K(Q)).
\end{aligned}$$

That is $D(K(P), K(Q)) = D(\imath K(P), \imath K(Q))$. \square

As a result of the above discussions and analyses, we come to the following three corollaries.

Corollary 1 Let $\mathbf{K}(U)$ be the set of all knowledge structures induced by U and $K(P), K(Q) \in \mathbf{K}(U)$ two knowledge structures. If $K(P) \preceq K(Q)$, then $D(K(P), K(\omega)) \leq D(K(Q), K(\omega))$.

Proof. From the knowledge structure $K(\omega) = \{x_i \mid x_i \in U\}$ and $K(P) \preceq K(Q)$, one has that $\{x_i\} \subseteq S_P(x_i) \subseteq S_Q(x_i)$, $x_i \in U$. Therefore,

$$\begin{aligned}
D(K(P), K(\omega)) &= \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|S_P(x_i) \oplus \{x_i\}|}{|U|} = \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|S_P(x_i)| - 1}{|U|} \\
&\leq \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|S_Q(x_i)| - 1}{|U|} = \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|S_Q(x_i) \oplus \{x_i\}|}{|U|} \\
&= D(K(Q), K(\omega)),
\end{aligned}$$

i.e., $D(K(P), K(\omega)) \leq D(K(Q), K(\omega))$. \square

Corollary 2 Let $\mathbf{K}(U)$ be the set of all knowledge structures induced by U and $K(P), K(Q) \in \mathbf{K}(U)$ two knowledge structures. If $K(P) \prec K(Q)$, then $D(K(P), K(\delta)) > D(K(Q), K(\delta))$.

Proof. Since the knowledge structure $K(\delta) = \{S_P(x_i) \mid S_P(x_i) = U, x_i \in U\}$ and $K(P) \preceq K(Q)$, so $S_P(x_i) \subseteq S_Q(x_i) \subseteq U$, $x_i \in U$. Hence,

$$\begin{aligned}
D(K(P), K(\delta)) &= \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|S_P(x_i) \oplus U|}{|U|} = \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|U| - |S_P(x_i)|}{|U|} \\
&\geq \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|U| - |S_Q(x_i)|}{|U|} = \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|S_Q(x_i) \oplus U|}{|U|} \\
&= D(K(Q), K(\delta)),
\end{aligned}$$

That is $D(K(P), K(\delta)) \geq D(K(Q), K(\delta))$. \square

Corollary 3 Let $\mathbf{K}(U)$ be the set of all knowledge structures induced by U and $K(P)$ a knowledge structure on $\mathbf{K}(U)$, then $D(K(P), K(\delta)) + D(K(P), K(\omega)) = 1 - \frac{1}{|U|}$.

Proof. Since $K(\omega) \preceq K(P) \preceq K(\delta)$, one can obtain that

$$D(K(P), K(\delta)) + D(K(P), K(\omega))$$

$$\begin{aligned}
&= \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|S_P(x_i)|-1}{|U|} + \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|U|-|S_P(x_i)|}{|U|} \\
&= \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|S_P(x_i)|-1+|U|-|S_P(x_i)|}{|U|} \\
&= \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|U|-1}{|U|} \\
&= 1 - \frac{1}{|U|}.
\end{aligned}$$

Obviously, $D(K(P), K(\delta)) + D(K(P), K(\omega)) = 1 - \frac{1}{|U|}$. \square

Remark. Unlike information entropy and knowledge granulation, the knowledge distance can characterize the difference between two knowledge structures in knowledge bases. From the definition of knowledge distance, it is easy to see that it is valid for complete knowledge structures and incomplete knowledge structures. It has some potential applications. For example, based on the knowledge distance between the knowledge structure induced by each condition attribute and the knowledge structure induced by the decision attribute, one can construct a heuristic function to extract decision rules with much higher certainty from a complete/incomplete decision table. Further experimental analysis may be desirable, but it is beyond the scope of this paper.

6. Conclusions and discussions

The contributions of the paper have three hands. In this paper, by uniformly representing a complete knowledge structure and an incomplete knowledge structure, firstly, we have proposed four operators (\cup , \cap , \wr and $-$) on a knowledge base, which can be applied to generate new knowledge structures. For a decision problem in the context of decision tables, these four operators can be used to extract decision rules with much higher certainty from a given decision table. Then, in this framework of knowledge representation proposed in this study, we have established an axiom definition of knowledge granulation and have proved that some existing knowledge granulations are all its special forms. The analysis shows that one can apply this axiom definition to construct a new knowledge granulation, which can be used to restrict a new definition of knowledge granulation according to practical demands and user requirements. Finally, we have introduced the definition of a knowledge distance for calculating the difference between two knowledge structures in the same knowledge base. Noting that the knowledge distance satisfies the three properties of a distance space on all knowledge structures induced by a given universe and $(\mathbf{K}(U), D)$ is a distance space. The knowledge distance can be used to distinguish the difference between two knowledge structures with the same knowledge granulation and to characterize the essence of uncertainty of knowledge structures in knowledge bases. These results have been shown to be very helpful for knowledge discovery from knowledge bases and significant for establishing a framework of granular computing in knowledge bases.

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